

The Isotropy of Compact Universes

John D. Barrow¹ and Hideo Kodama²

¹DAMTP, Centre for Mathematical Sciences,
Wilberforce Rd., Cambridge CB3 0WA, UK

²Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, Japan

February 7, 2008

Abstract

We discuss the problem of the stability of the isotropy of the universe in the space of ever-expanding spatially homogeneous universes with a compact spatial topology. The anisotropic modes which prevent isotropy being asymptotically stable in Bianchi-type VII_h universes with non-compact topologies are excluded by topological compactness. Bianchi type V and type VII_h universes with compact topologies must be exactly isotropic. In the flat case we calculate the dynamical degrees of freedom of Bianchi-type I and VII_0 universes with compact 3-spaces and show that type VII_0 solutions are more general than type I solutions for systems with perfect fluid, although the type I models are more general than type VII_0 in the vacuum case. For particular topologies the 4-velocity of any perfect fluid is required to be non-tilted. Various consequences for the problems of the isotropy, homogeneity, and flatness of the universe are discussed.

1 Introduction

The problem of providing a compelling explanation for the isotropy and approximate flatness of the Universe has been the subject of extensive analysis ever since the discovery of the temperature isotropy of the microwave background radiation. Historically, a variety of approaches have been taken to

solving these problems. The first was to justify choosing special initial conditions, at $t = 0$ or $t = -\infty$, or 'almost' initial conditions imposed at the Planck epoch $t \sim 10^{-43}s$ which marks the threshold of quantum gravity. The second was to seek out general physical processes which might transform a wide range of initial conditions into a state that is similar to the presently observed universe after billions of years of expansion (Misner 1967). This strategy was pursued first within the context of the 'chaotic cosmology' programme during the period 1967-80. The rules usually imposed were that classical dissipative stresses should obey the strong energy condition of general relativity. Typical scenarios considered involved the classical dissipation of anisotropy by collisional or collisionless transport processes, and the depletion of irregularities by the quantum particle production process (Doroshkevich, Zeldovich & Novikov 1968, Stewart 1968, Collins & Stewart, 1971, Zeldovich and Starobinskii 1972, Barrow 1977, Barrow & Matzner 1981).

One mathematically well-defined way of approaching the question of the naturalness of the isotropy of the universe was to investigate the stability of the isotropic Friedmann universes with respects to the set of all anisotropic solutions of a given gravitation theory. This was first done by Collins and Hawking (1973) in the context of spatially homogeneous general relativistic cosmologies with zero cosmological constant by considering the stability of non-collapsing isotropic universes in the space of Bianchi type universes with the natural \mathbb{R}^3 topology. This reduces to the study of the stability of particular solutions of ordinary differential equations but is made non-trivial by the appearance of eigenvalues with vanishing real part, and so the stability is generally determined at non-linear order in the expansion around the isotropic universes in general.

The most general Bianchi universes containing the open Friedmann universe as a special case are the Bianchi type VII_h spaces. The open Friedmann solution was shown by Collins and Hawking not to be asymptotically stable in the space of all VII_h initial data if matter obeyed the strong energy and positive density conditions. A more detailed investigation (Barrow 1982, Barrow & Sonoda 1986) revealed that as time $t \rightarrow \infty$ isotropy is stable (but not asymptotically stable) and identified the attractor as a family of exact vacuum plane-wave spacetimes found by Lukash (1974). It contains the isotropic Milne model as a special case.

The original result of Collins and Hawking is widely cited as showing that isotropy was unstable (see, for example, Kolb and Turner 1990). However, the asymptotic stability analysis of open universes is somewhat deceptive

because as $t \rightarrow \infty$ the imposition of the strong energy condition ensures that these models approach vacuum (curvature-dominated) solutions. Thus the asymptotic behaviour of the vacuum solutions ultimately determines the asymptotic stability properties of the isotropic non-vacuum solutions. However, our past cosmological history contains very little time (if any at all) during which vacuum behaviour could have dominated the Hubble expansion. Thus the asymptotic stability theorems are dictated by conditions which have not existed in our past for any significant time interval. They mainly tell us about the future. Thus, even if isotropy were an asymptotically stable property of open sets of Bianchi VII_h initial data, it would not explain the isotropy of our universe, because it would tell us only that isotropy is approached during the late curvature-dominated phase of expansion at redshifts less than $z \sim 10$.

If the strong energy condition is violated then the conclusions will change. Vacuum stresses of the sort that can drive inflation must violate the strong energy condition in order to accelerate the mean scale factor of the universe and isotropy can become asymptotically stable in accordance with cosmic no hair theorems (Barrow 1982, Barrow & Sonoda 1986). A particular case of this result would be the asymptotic approach to the de Sitter metric if a positive cosmological constant was admitted. Violation of the strong energy condition is necessary but not sufficient for isotropization to occur in this way.

If homogeneous initial conditions are confined to those in an open neighbourhood of the spatially flat Friedmann universe then Collins and Hawking (1973) went on to show that, in the space of Bianchi type VII_o initial data containing the flat Friedmann model as a special case, isotropy is asymptotically stable if the matter content is restricted to have zero pressure ('dust') to first order. The condition for isotropization requires $\sigma/H \rightarrow 0$ as $t \rightarrow \infty$, where σ is the expansion shear and H is the mean Hubble expansion rate. Isotropy is not approached during in radiation-dominated models of this Bianchi type. Thus if there was some reason for the initial data to be of zero curvature ('flat') it appeared that isotropy might be regarded as an asymptotically stable cosmological feature. However, again, the observational consequences are limited because there have been so many more e-foldings of cosmic expansion during an era of radiation domination (from $10^{-43}s$ to $10^{10}s$) than one dominated by dust (from $10^{10}s$ to $10^{17}s$) during the history of the universe (Barrow 1982; Barrow & Sonoda, 1985, 1986). Also, as in the open universe case, if the strong energy condition and zero

pressure condition restrictions are relaxed to permit the inclusion of stresses with $\rho + 3p < 0$, where ρ is the matter density and p is the pressure then isotropy becomes asymptotically stable for an open set of initial data. This situation has been investigated in detail by Wainwright et al (1999) and Nilsson et al (2000) who discriminate between isotropization of the expansion shear, σ , and that of the Weyl curvature anisotropy relative to the mean Hubble expansion rate, H . Radiation and dust-dominated VII_0 universes exhibit isotropic shear isotropization, $\sigma/H \rightarrow 0$ as $t \rightarrow \infty$ but the Weyl curvature anisotropy, W/H grows in the radiation dominated era and approaches a non-zero constant (which need not be small) in the dust era as $t \rightarrow \infty$ (Collins and Hawking did not require the Weyl curvature anisotropy to tend to zero asymptotically in order for isotropization to occur in type VII_0). This feature was also stressed by Doroshkevich et al (1973) and Lukash (1974). Unlike the shear anisotropy, a large Weyl anisotropy does not require a large temperature anisotropy in the microwave background. The type VII_h asymptote does have $W/H \rightarrow 0$ as $t \rightarrow \infty$ because it is a plane-wave spacetime for which all scalar curvature invariants are zero.

These results were interpreted as showing that if the strong energy condition holds (no inflation allowed) then generic spatially homogeneous ever-expanding anisotropic universes do not become isotropic during their expansion histories. A separate analysis of closed Bianchi type IX universes with S^3 topology shows that isotropy is unstable under the similar conditions (Doroshkevich et al 1973). Of course, if the strong energy condition is dropped then a finite period of inflationary expansion in the past can reduce any initial anisotropy observed inside our horizon below any given level by a pre-specified time and drive the expansion of any open or not too closed universe very close to flatness for very long intervals of time at late times without either isotropy or flatness being approached as $t \rightarrow \infty$. If it is too closed it may collapse before inflation can occur. Moreover, if further conditions of physical reality are imposed, so that the anisotropy energies in long-wavelength gravitational wave degrees of freedom at the Planck epoch do not significantly exceed the Planck density then the size of anisotropies today is constrained to be quite close to that observed in the microwave background (Barrow 1995).

The basis of these results about the stability of isotropic universes when the strong energy condition holds is the instability of isotropic universes to perturbations by spatially homogeneous anisotropic modes. The most general Bianchi type universes are characterized by 4 constant parameters on

a Cauchy surface of constant time in vacuum and by 8 parameters in the presence of a perfect fluid (Ellis and MacCallum 1969). The most general Bianchi types containing the flat, open, and closed Friedmann universes are of Bianchi types VII_0 , VII_h , IX , respectively and are specified by 3, 4, 4 constant parameters in vacuum and 7, 8, 8 parameters in the perfect fluid case, respectively. For comparison, in the presence of a perfect fluid, the flat Friedmann universe has no free parameters, and the open and closed models have 1 free parameter.

All these results assume that the topology of ever-expanding universes is the natural \mathbb{R}^3 topology so that they have infinite spatial volume if the 3-curvature is never positive. We shall show that the conclusions change significantly if it is assumed that the topology of the spatial sections of spatially homogeneous anisotropic universes of non-positive curvature are *spatially compact*.

2 Spatially Compact Bianchi type universes

There has been considerable interest in the possibility that the Universe might possess compact space sections of non-positive curvature (see the conference proceedings collection edited by Starkman1998). These considerations have been motivated in part by the 'naturalness' of finite spatial sections in quantum cosmologies. There is a long history of occasional investigations of the observational consequences of compact flat universes with 3-torus topology (Lachieze-Rey and Luminet 1995) and the possibility was even considered by Friedmann (1924) to show that open universes were not necessary spatially infinite. Studies of these compact cosmologies have been dominated by studies of multiple imagery (Sokolov and Shvartsman 1975) but there has been also some work which related the observed homogeneity and isotropy of the universe to its spatial topology. For example, the possibility of invoking a 'small' compact open or flat universe to account for the large scale uniformity of the universe has been proposed by Ellis and Schreiber (1986) following the pioneering consideration of topological effects on local cosmological structure by Ellis (1971). They have explored the possibility that the cosmological horizon problem can be significantly ameliorated by identifying points in such a way that observers can see all the way round the universe after some given time. Although this construction is easy to achieve in flat Friedmann universes, it is necessary to explore its effects upon

a fully anisotropic and inhomogeneous universe. In a related study, Ellis and Tavakol (1994) have considered the effects of geodesic mixing in compact open universes on the propagation of microwave background photons after last scattering and the topological constraints on anisotropy damping by this process were considered by Reboucas et al (1998).

The rapid progress in observations of the cosmic microwave background is also providing a possibility to observe directly the topology of the universe. In particular, it has recently been recognized that the changed appearance of the microwave background sky pattern offers a sensitive probe of the spatial topology if the scale of periodicity is sufficiently close to the particle horizon scale (Levin et al, 1997, 1999, Cornish, Spergel and Starkman 1998). In the flat Friedmann universe there is no reason why these two scales should be similar. However, open universes provide a physical curvature scale which might be closely related to the overall periodicity scale. For this reason there has been much recent interest in the observational features of compact open Friedmann universes. Their behaviour is considerably more diverse than that of compact flat universes. Geodesics display chaotic divergences on negatively curved spaces and possible topologies are extremely complex, the eigenmode problem is unsolved in general, and their specifications are only partially understood.

Let us now consider an extension of the problem of 'why the universe is isotropic?' to the situation of compact, spatially homogeneous, anisotropic universes. We will consider the compactification of Bianchi type universes with zero and negative curvature. As in the investigation made by Collins and Hawking (1973) for the non-compact case, we first ask what are the most general Bianchi types which contain the flat and open Friedmann universes as particular cases. This problem has been studied by several authors (Ashtekar and Samuel 1991, Fagundes 1985, 1992, Kodama 1998, Koike et al 1993, Tanimoto et al 1997, 1997a).

If a group contains a subgroup Γ which acts on a manifold X as a covering group so that X/Γ becomes compact then the geometry of the manifold is said to admit a compact quotient. Thurston (1979) classified all maximal simply-connected three-dimensional geometries which admit a compact quotient into one of eight possible cases.

The Bianchi classification of spatially homogeneous universes is derived from that of the three-dimensional Lie groups that act freely on a four-dimensional spacetime manifold as an isometry group with spatial orbits. We say that a homogeneous manifold is simply homogeneous if the group

of isometries has a three-dimensional subgroup that acts simply transitively on the manifold. The Bianchi types are subdivided into two classes (Ellis and MacCallum, 1969): Class A contains types $I, II, VI_0, VII_0, VIII$ and IX while Class B contains types IV, V, III, VI_h and VII_h . Except for types IV and VI_h , we can construct a spacetime manifold with compact space for these models by suitable identifications of their spatially homogeneous hypersurfaces. This operation will usually lower the dimension of the isometry group so that the spatial hypersurfaces of the resulting spacetime is no longer simply homogeneous. For example, it is known that a homogeneous space section in any Class B model cannot be simply homogeneous if it is compact. Compact quotients are often not homogeneous globally at all, (i.e., the full symmetry group does not act transitively). This happens even for the simple manifold T^3/\mathbb{Z}_2 which is a quotient of the Euclidean space E^3 by a discrete group generated by two translations in the $x - y$ plane and a combination of a translation along the z -axis and a rotation by angle π around the same axis. Therefore, in general, the spacetime with compact space can only be locally homogeneous. The same argument applies to isotropy as well.

Further new features appear in the compact cases. First, a non-trivial compact topology may require the geometry and matter configurations to have a higher symmetry than that of a simply transitive group when the data is lifted to the covering spacetime. Such a situation arises when the identification group Γ cannot be contained in a simply transitive group. This feature has an important consequence for the isotropization problem, as we will see below. Second, new dynamical degrees of freedom, called 'moduli parameters', appear. These moduli parameters describe globally non-isometric deformations of the geometry which preserve the local geometric structure. For some cosmological models, the total number of dynamical degrees of freedom becomes much larger than that for the non-compact case due to the existence of the moduli degrees of freedom.

2.1 Open universes

The introduction of compactness imposes a major constraint upon homogeneous anisotropic universes with negative spatial curvature. The Bianchi types V and VII_h contain open Friedmann universes as special cases. By inspecting the subgroup structure of each maximal symmetry group, one finds that these groups must be subgroups of the maximal symmetry corresponding to the Thurston type H^3 if they act simply transitively on constant-time

spatial sections $\tilde{\Sigma}$ of the covering spacetime. Hence, by Thurston's theorem, if the universal covering (\tilde{M}, \tilde{g}) of a locally homogeneous spacetime (M, g) with compact space has a symmetry group G containing a simply transitive subgroup of Bianchi type V or VII_h , then G must be a subgroup of the isometry group of H^3 , and so (M, g) is written as $(\tilde{M}, \tilde{g})/\Gamma$ with some discrete group $\Gamma \subset G$.

It is here, when determining minimal possible G , that the Class A or B nature of the homogeneity group becomes important. For the Class B Bianchi space sections, $\tilde{\Sigma}$, there exists a non-vanishing vector field v written as $q^{IJ}c_I X_J$ where X_I is an invariant basis with a structure constant c^I_{JK} , $c_I = c^J_{IJ}$ and q_{IJ} is the component of a homogeneous spatial metric with respect to X_I . Since this vector has non-vanishing divergence, $\nabla \cdot v = g^{IJ}c_I c_J$, it cannot be invariant under the action of Γ . For, if it were invariant, it would create a well defined vector field v' with non-vanishing divergence on the quotient $\Sigma = \tilde{\Sigma}/\Gamma$. But this would lead to a contradiction since

$$0 = \int_{\Sigma} d^3x \sqrt{g} \nabla \cdot v' = \nabla \cdot v' \int_{\Sigma} d^3x \sqrt{g} \neq 0.$$

However, one finds that if G is smaller than the connected component of the full isometry group of H^3 , G keeps v invariant. Therefore the covering spacetime must be invariant under a group isomorphic to the maximal isometry group of H^3 and be spatially homogeneous and isotropic.

Furthermore there is no moduli freedom in the Class B case from the Mostow rigidity theorem (Thurston 1979, 1982), which says that two compact hyperbolic spaces are isometric up to a constant scaling if they are homeomorphic. Thus compact universes of Bianchi types V and VII_h admit only an overall change of volume scaling factor. They must therefore be isotropic. Hence, we have the rather surprising result that compact open universes of type VII_h not only make isotropy an asymptotically stable property of the initial data set but they permit no anisotropy to be present at all. Recall that in the non-compact stability analyses of the open Friedmann models the asymptotic behaviour was approach to a particular 2-parameter set of anisotropic type VII_h plane-wave universes in which the ratio of the shear to Hubble scalar is a constant (when this ratio is zero the isotropic vacuum Milne universe is obtained). These anisotropic solutions are not admitted when the space sections are compactified. Compactification requires

that Bianchi type initial data containing isotropic universes must be exactly isotropic.

2.2 Flat universes

The flat Friedmann universe is a special case of Bianchi types I and VII_0 . In these Class A universes the introduction of compactification does not restrict the symmetry group so strongly as in the Class B cases, and anisotropy is permitted. However, the spatial topology still gives some weak constraints on the minimal symmetry of the universal covering as shown in Table 1. For example, when the spatial sections are homeomorphic to T^3 , T^3/\mathbb{Z}_2 or $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, under an appropriate choice of coordinates, the minimal symmetry of the covering data with the Bianchi I symmetry is given by $\mathbb{R}^3 \tilde{\times} D_2$, while for T^3/\mathbb{Z}_k ($k = 3, 4, 6$) the minimal symmetry is $\mathbb{R}^3 \tilde{\times} O(2)$. Here, D_2 is the dihedral group $\{1, R_1(\pi), R_2(\pi), R_3(\pi)\}$ consisting of rotations by angle π around three orthogonal coordinate axes; $O(2)$ is the group generated by rotations around the z -axis and D_2 , and $A \tilde{\times} B$ is a semi-direct product of two groups A and B . Hence, the isotropization behavior of the former group is the same as that of the generic open type I case, but the behavior for the latter group coincides with that of axisymmetric type I model. However, isotropy is an asymptotically stable property for spatially compact Bianchi I models with perfect fluid satisfying the dominant energy condition. The Bianchi type I geometry does not permit the presence of non-comoving velocities.

Similarly, for T^3 and T^3/\mathbb{Z}_2 , the minimal symmetry of the covering data with type VII_0 symmetry coincides with the VII_0 group itself, hence the isotropization behavior is determined by that of generic open type VII_0 models with non-vanishing generic fluid velocity. In contrast, for $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ and T^3/\mathbb{Z}_k ($k = 3, 4, 6$), the minimal symmetry is $VII_0 \tilde{\times} D_2$, which requires that the space metric be diagonal and the fluid 4-velocity thus be orthogonal to the spacelike hypersurfaces of constant time. Since the isotropic type VII_0 model with radiation is already unstable against perturbations with the latter symmetry, (Wainwright et al 1999, Nilsson et al 2000) compactification does not change the conclusion on the isotropization of type VII_0 Bianchi models.

| Symmetry | Space | Q | P | N_m | N_f | N | N_s | $N_s(\text{vacuum})$ |
|--|--|-----|-----|-------|-------|-----|-------|----------------------|
| $\mathbb{R}^3 \tilde{\times} D_2$ | \mathbb{R}^3 | 0 | 3 | 0 | 1 | 4 | 2 | 1 |
| | T^3 | 3 | 3 | 6 | 1 | 13 | 11 | 10 |
| | T^3/\mathbb{Z}_2 | 3 | 3 | 2 | 1 | 9 | 7 | 6 |
| | $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ | 3 | 3 | 0 | 1 | 7 | 5 | 4 |
| VII_0 | \mathbb{R}^3 | 2 | 3 | 0 | 4 | 9 | 7 | (3) |
| | T^3 | 3 | 3 | 4 | 4 | 14 | 12 | (8) |
| | T^3/\mathbb{Z}_2 | 3 | 3 | 2 | 4 | 12 | 10 | (6) |
| $\text{VII}_0 \tilde{\times} \mathbb{Z}_2$ | \mathbb{R}^3 | 2 | 3 | 0 | 2 | 7 | 5 | (3) |
| | T^3 | 3 | 3 | 4 | 2 | 12 | 10 | (8) |
| | T^3/\mathbb{Z}_2 | 3 | 3 | 2 | 2 | 10 | 8 | (6) |
| $\text{VII}_0 \tilde{\times} D_2$ | \mathbb{R}^3 | 2 | 3 | 0 | 1 | 6 | 4 | 3 |
| | T^3 | 3 | 3 | 4 | 1 | 11 | 9 | 8 |
| | T^3/\mathbb{Z}_2 | 3 | 3 | 2 | 1 | 9 | 7 | 6 |
| | $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ | 3 | 3 | 1 | 1 | 8 | 6 | 5 |
| | $T^3/\mathbb{Z}_k (k = 3, 4, 6)$ | 3 | 3 | 0 | 1 | 7 | 5 | 4 |
| $\mathbb{R}^3 \tilde{\times} O(2)$ | \mathbb{R}^3 | 0 | 2 | 0 | 1 | 3 | 1 | 0 |
| | T^3 | 2 | 2 | 4 | 1 | 9 | 7 | 6 |
| | $T^3/\mathbb{Z}_2(\text{aligned})$ | 2 | 2 | 2 | 1 | 7 | 5 | 4 |
| | $T^3/\mathbb{Z}_2(\text{oblique})$ | 2 | 2 | 3 | 1 | 8 | 6 | 5 |
| | $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ | 2 | 2 | 1 | 1 | 6 | 4 | 3 |
| | $T^3/\mathbb{Z}_k (k = 3, 4, 6)$ | 2 | 2 | 0 | 1 | 5 | 3 | 2 |
| $ISO(3)$ | \mathbb{R}^3 | 0 | 1 | 0 | 1 | 2 | 0 | 0 |
| | T^3 | 1 | 1 | 5 | 1 | 8 | 6 | 5 |
| | T^3/\mathbb{Z}_2 | 1 | 1 | 3 | 1 | 6 | 4 | 3 |
| | $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ | 1 | 1 | 2 | 1 | 5 | 3 | 2 |
| | $T^3/\mathbb{Z}_k (k = 3, 4, 6)$ | 1 | 1 | 1 | 1 | 4 | 2 | 1 |

Table 1: The number of dynamical degrees of freedom for locally homogeneous systems with perfect fluid on the spaces of Thurston type E^3 . N_s represents the dimension of the solution space. See the Appendix for the details of the notation.

| Bianchi Type | Vacuum | | Perfect fluid | |
|-------------------------|-------------|---------|---------------|---------|
| | Non-compact | Compact | Non-compact | Compact |
| <i>I</i> | 1 | 10 | 2 | 11 |
| <i>II</i> | 2 | 6 | 5 | 9 |
| <i>VI</i> ₀ | 3 | 4 | 7 | 8 |
| <i>VII</i> ₀ | 3 | 8 | 7 | 12 |
| <i>IX</i> | — | 4 | — | 8 |

Table 2: Maximal degrees of freedom for spatially open and compact Bianchi vacuum models and for those with perfect fluid.

2.3 Parameter Counting

A parameter-counting classification of Bianchi type universes can be performed in the compact case, as is explained in the Appendix in detail. Some of the constant parameters represent the moduli freedom, but in general it is difficult to distinguish clearly between the dynamical freedom and the moduli freedom. The general result of Kodama (1998), Koike et al (1993), and Tanimoto et al (1997, 1997a) is that the number of dynamical degrees of freedom, i.e., the dimension of the solution space, becomes larger in the compact cases than in the non-compact cases for the Class A Bianchi universes. This is partly due to the appearance of the moduli degrees of freedom and partly due to the decrease in the freedom of diffeomorphisms connecting physically equivalent solutions. For example, in the case of simple 3-torus, T^3 , we can obtain the correct parameter count for the vacuum case by the following naive argument. First, in order to specify the lattice in the Euclidean 3-space so as to define the 3-torus in a rotationally invariant way, we need 3 parameters to specify the lengths of 3 vectors generating the lattice and 3 parameters to specify the relative direction angles of these vectors. Hence, adding their time derivatives, we need 12 parameters. If we take into account the Hamiltonian constraint and the time translation freedom, the total number reduces to 10.

One significant feature of the parameter count in the spatially compact open universes is the difference between the counts for vacuum and perfect fluid models. If we separate the moduli and dynamical freedoms in the way explained in the Appendix, the diffeomorphism-invariant phase space can be, roughly speaking, written as the product of the phase space of the diagonal homogeneous system and the moduli space, at least for the vacuum Bianchi

I, II, VI_0 and VII_0 systems. We can show that the moduli parameters defined in this sense become constants of motion. Hence, the dynamics are essentially represented by the diagonal system with $3 + 3$ parameters. The Hamiltonian constraint should then also be imposed. In this representation, the phase space of the non-compact case is obtained by discarding the moduli freedom and taking into account the additional equivalence relation for the diagonal system. For example, in the type I case, 3 metric coefficients of the diagonal system can be set to unity by a change of spatial coordinates at the initial time, and we obtain a final count of $6 - 3 = 3$. The Hamiltonian constraint and the temporal gauge fixing reduces it further to $3 - 2 = 1$. In the compact case, we cannot do such a rescaling because it changes the moduli parameters, and the count is 10. From Table 1, we find that it is 8 for the vacuum VII_0 system on T^3 . This implies that among the spatially compact *vacuum* Bianchi models, the type I model is the most generic unlike in the usual situation with non-compact topology, where it is the least generic.

In contrast, when a perfect fluid is present, Bianchi type VII_0 universes need not be diagonalizable because the momentum constraints simply relate the non-diagonal components of the extrinsic curvature to the spatial components u_I of the fluid 4-velocity. Since the momentum constraints require u_I to vanish for type I models, it turns out that the parameter count for type VII_0 models is always larger than that for type I models in any given space topology. Therefore, the most general locally homogeneous *perfect-fluid* spacetimes that include the flat isotropic model are the Bianchi type VII_0 models in the spatially compact case just as they are in the non-compact case.

In the table we have also included, for comparison, the parameter counts for the compact Bianchi type IX universes with S^3 topology which contain the closed Friedmann universes as isotropic sub-cases. The vacuum IX universes are 4-parameter, while the perfect fluid type IX universes are 8-parameter.

Here note that it is easy to extend these parameter counts to systems with scalar fields. Since the homogeneity-group preserving diffeomorphisms act trivially on scalar fields and scalar fields do not contribute to the momentum constraint in a locally homogeneous spacetime, the parameter count simply increases by two for each real component of the scalar fields irrespective of the Bianchi type and the other matter contents. Hence, the compact Bianchi I models are still more general than the compact Bianchi VII models if the models contain only scalar fields.

When counting 'parameters' in the compact cases it is important to remember that the lengths and the angles of the identifications are time-dependent variables specifying the compact spatial geometry g_{ij} at each time. Since the Einstein equation is second order in time, we must also specify their time derivatives, \dot{g}_{ij} , at an initial time, and then their future time development is completely determined. For the simple 3-torus case, six combinations of these variables and their time derivatives become the moduli parameters, which turn out to be constant in time because the Hamiltonian constraint does not depend on them. The other six combinations correspond to the diagonalized metric components and their time derivatives. However, such a simple counting argument sometimes fails for other topologies because the canonical structure becomes degenerate in the moduli sector; that is, some moduli parameters do not have conjugate momenta, as was shown in Kodama (1998). Since the existence and the uniqueness up to diffeomorphisms of the generic initial value problem holds for the Einstein equations, we can determine the number of independent solutions even in such cases by calculating the dimension of the locally homogeneous sector of the full diffeomorphism-invariant phase space. This is the approach adopted in Kodama (1998), and differs from that adopted in the papers by Koike et al (1993), and Tanimoto et al, (1997, 1997a) in which spacetime solutions are classified. In the latter approach, knowledge on the explicit form of spacetime solutions is required.

Finally, note that in contrast to the cases where space has \mathbb{R}^3 topology, it is meaningless to compare the parameter counts for spatially compact Bianchi models belonging to different Thurston types such as E^3 , Nil (type *II*) and Sol (type VI_0), because spaces belonging to different Thurston types are not homeomorphic with each other.

2.4 Self-similar Bianchi types

It is of great interest to see if our results can be extended to inhomogeneous and anisotropic universes. In general this is a very difficult mathematical problem. One simple class of inhomogeneous relatives of the Bianchi type universes is provided by their self-similar extensions, first found by Eardley (1974). It can be seen that there are no self-similar spacetimes with compact spaces if the self-similar group is not an isometry group and acts simply transitively. For, as Eardley shows, in such a case the self-similarity group, H , contains an isometry group, G , with $\dim G = \dim H - 1$, and each orbit of G is two-dimensional. In the case of an open universe, this gives \mathbb{R}^3 the

structure of a fibre bundle on a line \mathbb{R} with fibres given by the orbits of G . If it can be compactified by some discrete group, Γ , then Γ must be contained in the isometry group G . Hence taking the quotient with Γ may compactify the fibres but it leaves the underlying line \mathbb{R} unchanged. Therefore the quotient space is at most a fibre bundle on a line with a compact 2-space, and cannot be a compact 3-space.

3 Discussion

We have shown that the stability of open and flat isotropic universes in the space of spatially homogeneous initial data is strongly affected by the global topology of the universe. When the topology is compact, all Class B Bianchi type universes are forbidden unless they are isotropic. In particular, this means that the universes of Bianchi type VII_h , and V which contain the open Friedmann universes must be isotropic.

The most general anisotropic universes containing the flat Friedmann universes are of Bianchi type VII_0 when the topology is non-compact. However, in the presence of a compact topology the most general flat anisotropic universes are of Bianchi type I in the vacuum case and of type VII_0 in the presence of a perfect fluid. Thus, in the perfect fluid case the stability properties of the isotropic models at late times isotropic models is the same as for universes with non-compact topologies with non-decay of anisotropic curvature distortions as $t \rightarrow \infty$ in the radiation and dust dominated solutions, as found in earlier studies.

One consequence of this new behaviour in compact spaces is to change the familiar conclusion that open universes are more general than flat universes because they require more parameters for the specification of their initial data. When homogeneous anisotropy is present the open universes are forbidden while flat universes are allowed. This may have important consequences for the assessment of the significance of the 'flatness' problem in universes with compact topologies. It also indicates that in compact open and flat universes there is a close link between the properties of isotropy and flatness. These links should be investigated further in the context of inhomogeneous cosmologies.

The local isotropy of Bianchi VII_h models with compact space implies that the isotropy problem is replaced by the homogeneity problem for inhomogeneous perturbations of isotropic universes. That is, if there are some

physical processes which homogenize a perturbed VII_h compact universe globally, they must automatically isotropize the universe because homogeneous anisotropic VII_h universes cannot exist. Alternatively, there may be strong restrictions on the possibility of inhomogeneous open universes with compact topologies. Our results also suggest the possibility that the degree of anisotropy of the universe is constrained by the degree of inhomogeneity if the universe is approximately described by a Friedmann metric with a compact space of negative curvature locally. Such constraint will apply even to inflationary universes and might be used to determine whether or not the universe is negatively curved when the present universe is almost flat due to inflation.

The strong impact of compactification upon the range of possible anisotropic and homogeneous open universes is surprising. It shows one of the ways in which topological restrictions can impose significant constraints on the deviation of solutions of Einstein's equations from the simplest isotropic cases that resemble the observed universe today. These results can be generalized to other gravity theories which extend general relativity by adding certain higher-order curvature terms to the lagrangian. These higher-order terms are typically negligible at late times in ever-expanding universes when the higher-order curvature scalars become smaller than the linear terms contributed by general relativity.

One possible interpretation of our results is that the simultaneous presence of anisotropy and spatial homogeneity is a very special combination. In the cases of open or flat universes with non-compact topologies, the specialness of this type of homogeneous anisotropy is disguised by the degrees of freedom that are permitted for the homogeneous anisotropies. However, the imposition of topological compactness is extremely restrictive and makes some Bianchi geometries impossible in many anisotropic configurations. This type of restriction was also evident when rotation was introduced into closed compact Bianchi type IX universes (Collins and Hawking 1973a, Barrow et al 1985). A very strong limit on cosmic vorticity is created by attempting to accommodate a complex rotational dynamics in a finite positively curved space. The constraints on cosmic vorticity are accordingly much weaker in non-compact flat and open universes.

Although the spatial compactness restricts anisotropy more weakly for the Bianchi models including the spatially flat Friedmann universe, some observable relations between anisotropy and inhomogeneity may exist for particular space topologies. For example, if the topology of the space is given

by one of $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ and T^3/\mathbb{Z}_k ($k = 3, 4, 6$), then tilting of the fluid velocity is forbidden if the universe is locally homogeneous. This suggests that there may be a constraint on the degree of tilting by the degree of inhomogeneity. Thus it will be interesting to investigate the relation between anisotropy and inhomogeneity in the framework of linear perturbation theory on spatially compact Bianchi models. There will also be similar restrictions on certain types of anisotropic stress, for example those contributed by magnetic and electric fields or by collisionless massless particles, in these topologies.

It is a challenging and important problem to extend our analysis to the case of inhomogeneous cosmologies. We found that this cannot be done for the self-similar extensions of the Bianchi universes classified by Eardley (1974). We know that inhomogeneous solutions of Einstein's equations that are open or flat, with non-compact topologies, have initial data specified on a spacelike hypersurface of constant time by 4 arbitrary functions of 3 spatial coordinates in the vacuum case and by 8 arbitrary functions in the perfect fluid case. In the spatially homogeneous case these arbitrary functions become arbitrary constants. While this suggests that the general inhomogeneous solution may have parts that look locally like small perturbations of the spatially homogeneous models, this need not be the case for flat or open universes with compact topologies. Our study shows that the topology of the universe can impose significant restrictions upon the type of anisotropies that it can sustain.

Acknowledgments

We would like to thank Gary Gibbons, Janna Levin, and Stephen Siklos for helpful discussions. JDB acknowledges support from the Royal Society, the PPARC, and the University of New South Wales. HK was supported by the Grant-In-Aid for the Scientific Research (C2) of the Ministry of Education, Science, Sports and Culture in Japan (11640273).

Appendix

In this appendix we explain how to count the number of independent parameters, N_s , specifying the diffeomorphism classes of solutions to the Einstein equations for a locally homogeneous system.

First, we briefly review how solutions to the Einstein equations for a locally homogeneous system are related to those for a homogeneous system with a simply connected space. Let $M = \Sigma \times \mathbb{R}$ be a locally homogeneous

| Symmetry | Space | Q | P | N_m | N_f | N | N_s | $N_s(\text{vacuum})$ |
|---|---|-----|-----|-------|-------|-----|-------|----------------------|
| II | \mathbb{R}^3 | 1 | 3 | 0 | 3 | 7 | 5 | (2) |
| | $T^3(n)$ | 4 | 4 | 0 | 3 | 11 | 9 | (6) |
| $\text{II} \tilde{\times} \mathbb{Z}_2$ | \mathbb{R}^3 | 1 | 3 | 0 | 2 | 6 | 4 | (2) |
| | $T^3(n)$ | 3 | 3 | 2 | 2 | 10 | 8 | (6) |
| | $K^3(n)$ | 3 | 3 | 0 | 2 | 8 | 6 | (4) |
| $\text{II} \tilde{\times} D_2$ | \mathbb{R}^3 | 1 | 3 | 0 | 1 | 5 | 3 | 2 |
| | $T^3(n)$ | 3 | 3 | 2 | 1 | 9 | 7 | 6 |
| | $K^3(n)$ | 3 | 3 | 0 | 1 | 7 | 5 | 4 |
| | $T^3(n)/\mathbb{Z}_2$ | 3 | 3 | 2 | 1 | 9 | 7 | 6 |
| | $T^3(n)/\mathbb{Z}_2 \times \mathbb{Z}_2$ | 3 | 3 | 0 | 1 | 7 | 5 | 4 |
| Isom(Nil) | \mathbb{R}^3 | 1 | 2 | 0 | 1 | 4 | 2 | 1 |
| | $T^3(n)$ | 2 | 2 | 2 | 1 | 7 | 5 | 4 |
| | $K^3(n)$ | 2 | 2 | 1 | 1 | 6 | 4 | 3 |
| | $T^3(n)/\mathbb{Z}_2$ | 2 | 2 | 2 | 1 | 7 | 5 | 4 |
| | $T^3(n)/\mathbb{Z}_2 \times \mathbb{Z}_2$ | 2 | 2 | 1 | 1 | 6 | 4 | 3 |
| | $T^3(n)/\mathbb{Z}_k (k = 3, 4, 6)$ | 2 | 2 | 0 | 1 | 5 | 3 | 2 |

Table 3: The number of dynamical degrees of freedom for locally homogeneous systems with perfect fluid on the spaces of Thurston type Nil.

| Symmetry | Space | Q | P | N_m | N_f | N | N_s | $N_s(\text{vacuum})$ |
|--|------------------------|-----|-----|-------|-------|-----|-------|----------------------|
| VI_0 | \mathbb{R}^3 | 2 | 3 | 0 | 4 | 9 | 7 | (3) |
| | $\text{Sol}(n)(n > 2)$ | 3 | 3 | 0 | 4 | 10 | 8 | (4) |
| $\text{VI}_0 \tilde{\times} \{1, R_3(\pi)\}$ | \mathbb{R}^3 | 2 | 3 | 0 | 2 | 7 | 5 | (3) |
| | $\text{Sol}(n)$ | 3 | 3 | 0 | 2 | 8 | 6 | (4) |
| $\text{VI}_0 \tilde{\times} \{1, J\}$ | \mathbb{R}^3 | 2 | 3 | 0 | 2 | 7 | 5 | (3) |
| | $\text{Sol}(n)(n > 2)$ | 3 | 3 | 0 | 2 | 8 | 6 | (4) |
| $\text{Isom}^+(\text{Sol})$ | \mathbb{R}^3 | 2 | 3 | 0 | 1 | 6 | 4 | 3 |
| | $\text{Sol}(n)$ | 3 | 3 | 0 | 1 | 7 | 5 | 4 |

Table 4: The number of dynamical degrees of freedom for locally homogeneous systems with perfect fluid on the spaces of Thurston type Sol. J is a discrete transformation which permutes two of the invariant basis.

spacetime, and $\tilde{M} = \tilde{\Sigma} \times \mathbb{R}$ be its universal covering. The metric g and the matter configuration Φ on M can be lifted to \tilde{M} . The manifolds M and \tilde{M} are related by $M = \tilde{M}/\Gamma$, where Γ is a discrete group of transformations on \tilde{M} which preserve each constant time slice, $\tilde{\Sigma}(t) = \tilde{\Sigma} \times \{t\}$. This lift $(\tilde{g}, \tilde{\Phi})$ must be invariant under the action of Γ and, from the assumption of local homogeneity, Γ is included in a larger group, G , which acts simply homogeneously on \tilde{M} .

In the synchronous coordinates on $\tilde{M} = \tilde{\Sigma} \times \mathbb{R} \ni (x, t)$, the action of G can be expressed as time-independent transformations of $\tilde{\Sigma}$. Let G' be a subgroup of G which acts simply transitively on $\tilde{\Sigma}$, and let $\chi^0 = dt$ and $\chi^I (I = 1, 2, 3)$ be the invariant basis on $\tilde{\Sigma}$ with respect to G' . Then the lifted data on \tilde{M} , when expressed as components with respect to the invariant basis, is specified by a set of functions of time $X(t)$, which obey a set of first-order autonomous ordinary differential equations with four constraints, corresponding to the Hamiltonian constraint and the three momentum constraints. In the vacuum case, $X(t)$ is given by the spatial metric components Q_{IJ} and their conjugate momenta P^{IJ} . In the perfect-fluid case we must also include the fluid density

ρ and the three spatial components of the fluid velocity, u_I . If G is larger than G' , then X must have additional symmetry. Since Γ is included in G , these data are automatically invariant under Γ . Hence, for each solution $X(t)$ to the Einstein equations on \tilde{M} , the pair $(X(t), \Gamma)$ determines a solution to the Einstein equations on M . The functions $X(t)$ are also uniquely determined by the initial data $X(t_0)$ at some time $t = t_0$.

All the solutions on M are obtained in this way. However, two solutions on M derived from $(X_1(t_0), \Gamma_1)$ and $(X_2(t_0), \Gamma_2)$ may be connected by a diffeomorphism on M , and so be physically equivalent. It can be shown that this happens when and only when the two solutions are related by a time translation, or the two initial data are connected as $X_2(t_0) = f_* X_1(t_0)$ and $\Gamma_2 = f \Gamma_1 f^{-1}$ by a so called *homogeneity-group-preserving diffeomorphism* (HPD) f which preserves the symmetry group G in the sense that $f G f^{-1} = G$. All the HPDs form a group, $\text{HPDG}(G)$, which is the normalizer group of G in $\text{Diff}(\tilde{M})$ in mathematical terminology.

This gauge freedom can be removed to produce a unique specification of each diffeomorphism class in the following way. First, we introduce the moduli parameters as a coordinate system of a maximal submanifold \mathcal{M} in the space of Γ which is transversal to orbits of the action of $\text{HPDG}(G)$. If the isotropy group H of the action of $\text{HPDG}(G)$ is trivial, the gauge freedom is completely removed by this procedure. On the other hand, if the isotropy group is non-trivial, we further introduce a set of parameters as a coordinate system on a submanifold in the space of Q_{IJ} which is transversal to orbits of the H action. If H still has a non-trivial isotropy group, we further apply the same procedure to other variables, say P^{IJ} . Eventually, we obtain a set of parameters specifying the diffeomorphism classes of initial data for the locally homogeneous system, i.e., a coordinate system for the phase space $\Gamma_{\text{inv}}(M, G)$. The number of parameters classifying the diffeomorphism classes of solutions is obtained by subtracting 1 from the dimension of $\Gamma_{\text{inv}}(M, G)$.

This method can be also applied to the standard spatially homogeneous system with a simply connected space by considering the case in which Γ is trivial. For example, in the vacuum Bianchi I system, X is given by (Q_{IJ}, P^{IJ}) , and $\text{HPDG}(\mathbb{R}^3)$ is given by $IGL(3, \mathbb{R})$ whose element induces a similar transformation of the matrices Q and P . By HPDG Q can be set to the unit matrix. The isotropy group acts on P as $O(3)$. Hence P can be put into a diagonal form. After this reduction, the action of the residual HPDG on the initial data becomes trivial. Hence, by taking account of the Hamiltonian constraint and the time translation freedom, we find that the

equivalence class of the solutions are specified by a single parameter, which will correspond to one of the Kasner indices.

As a procedure to determine N_s for non-compact Bianchi models, our method is more complicated than those used by Siklos (1978) or by Ellis and MacCallum (1969). In their methods Γ_{inv} is simply obtained as $S/GL(3)$ or $S/SO(3)$, where S is the space of components with respect to a general invariant basis and its structure constant, $(Q_{IJ}, P^{IJ}, C^I_{JK})$, and that with respect to an orthonormal invariant basis, (P^{IJ}, C^I_{JK}) , respectively. In contrast, in our method, we need information on HPDG, which is in general obtained by a long calculation. However, in calculating N_s for compact locally homogeneous system, this information is indispensable to determine the moduli degrees of freedom correctly.

Tables 1–4 summarize the number of dynamical degrees of freedom obtained by this procedure for locally homogeneous systems with perfect fluid on compact spaces with Thurston type E^3 , Nil and Sol as well as the corresponding ones for Bianchi types I, VII_0, II , and VI_0 on \mathbb{R}^3 . In these tables, Q, P, N_m and N_f denote the number of independent variables to specify the space metric, the extrinsic curvature, the moduli freedom and the fluid freedom respectively, in the diffeomorphism invariant phase space $\Gamma_{\text{inv}}(M, \tilde{G})$ with an invariance group of the data on a space type M . N and N_s represent the dimensions of $\Gamma_{\text{inv}}(M, \tilde{G})$ and of the solution space. For comparison, N_s for vacuum systems are given on the final column where $(*)$ implies that the corresponding vacuum system has a higher discrete symmetry. Note that the total number of dynamical degrees of freedom in this table does not take the Hamiltonian constraint into account, and hence is greater by 1 than the dimension of $\Gamma_{\text{inv}}(M, G)$ defined above. It should be also noted that in the case of Bianchi type I , the lowest symmetry is not \mathbb{R}^3 and there are additional discrete symmetries $D_2 = \{1, R_1(\pi), R_2(\pi), R_3(\pi)\}$ corresponding to rotations by angle π around each coordinate axis because the fluid velocity is obliged to be orthogonal to constant time hypersurfaces due to the momentum constraints. If this D_2 symmetry exists, the spacetime metric can be put into diagonal form with respect to the time-independent invariant basis. In contrast, if the symmetry group G does not contain D_2 , such time-independent diagonalization is not possible even if Q and P have three dynamical degrees of freedom respectively. In such cases, the diagonalization of the metric requires time-dependent HPDs, which produce a non-vanishing shift vector.

If one allows for the time-dependent HPDs, one can always put the vari-

ables X into the form for the system on \mathbb{R}^3 with the same symmetry. This transformation transfers some degrees of freedom in Q and P back into the moduli freedom and makes the moduli parameters time-dependent. This enlarged moduli freedom is often used as the definition of moduli freedom in the literature. However, the description of dynamics becomes more complicated using this approach.

References

- [1] C. Misner, 1967, Phys. Rev. Lett. 19, 533
- [2] Y.B. Zeldovich and A.A. Starobinskii, Sov. Phys. JETP 34, 1159
- [3] J.D. Barrow, Nature 1977. Nature, 267, 117-120
- [4] J.D. Barrow and R. A. Matzner, 1981, Mon. Not. R. astron. Soc. 181, 719
- [5] J.M. Stewart, 1968, Astrophys. Letts. 2, 133
- [6] C.B. Collins and J.M. Stewart, 1971, Mon. Not. R. astron. Soc. 153, 419
- [7] C.B. Collins and S.W. Hawking, 1973, Astrophys. J. 180, 317
- [8] C.B. Collins and S.W. Hawking, 1973a, Mon. Not. R. astron. Soc. 162, 307
- [9] J. Wainwright, M.J. Hancock, and C. Uggla, 1999, Class. Quantum Grav. 16, 2577
- [10] U.S. Nilsson, M.J. Hancock and J. Wainwright, 2000, Class. Quantum Grav. 17, 3119.
- [11] A. Doroshkevich, Y.B. Zeldovich and I.D. Novikov, 1968, Sov. Phys. JETP 26, 408
- [12] J.D. Barrow and F.J. Tipler, *The Anthropic Cosmological Principle*, Oxford UP (1986)
- [13] A.D. Doroshkevich, V.N. Lukash and Novikov, I.D., 1973, Sov. Phys. JETP 37, 739

- [14] V.N. Lukash, 1975, Sov. Phys. JETP 40, 792
- [15] E.W. Kolb and M.S. Turner, *The Early Universe*, Addison Wesley, NY, (1990)
- [16] J.D. Barrow, 1982, Quart. Jl. Roy. astr. Soc., 23, 344
- [17] J.D. Barrow and D.H. Sonoda, 1986, Phys. Reports, 139, 1
- [18] J.D. Barrow and D.H. Sonoda, 1985, Gen. Rel. Gravn., 17, 409
- [19] J.D. Barrow, Phys. Rev. D 1995. Phys. Rev. D51, 3113.
- [20] G.F.R. Ellis and M.A.H. MacCallum, 1969, Comm. Math. Phys. 12, 108
- [21] G. Starkman, Class.Quant.Grav. 15, 2529 (1998)
- [22] M. Lachieze-Rey and J.P. Luminet, 1995, Phys. Reports 254,135
- [23] D.D. Sokolov and V.F. Shvartsman, 1975, Sov. Phys. JETP 39, 196
- [24] A.A. Friedmann, 1924, Zeit. f. Physik, 21, 326.
- [25] G.F.R. Ellis, 1971, Gen. Rel. Gravitation 2, 7.
- [26] G.F.R. Ellis and G. Schreiber, 1986, Phys. Lett. A 115, 97.
- [27] G.F.R. Ellis and R. Tavakol, 1994, in Deterministic Chaos In General Relativity, eds. D. Hobhill, A. Burd, and A. Coley, Plenum, New York, pp. 237-250.
- [28] M. Reboucas, R. Tavakol and A. Teixeira, 1998, Gen. Rel. Gravitation 30, 535.
- [29] J. Levin, J.D. Barrow, E. Bunn, and J. Silk, 1997. Phys. Rev. Lett. 79, 974
- [30] J. Levin, E. Scannapieco, G. De Gasperis, J. Silk and J.D. Barrow 1999. Phys. Rev. D 58, 123006-1
- [31] N. Cornish, D.N. Spergel and G.D. Starkman, 1998, Class. Quantum Grav. 15, 2657
- [32] H.V. Fagundes, 1985, Phys. Rev. Lett. 54, 1200

- [33] H.V. Fagundes, 1992, Gen. Rel. Grav 24, 199
- [34] H. Kodama, 1998, Prog. Theor. Phys. 99, 173-236
- [35] T. Koike, M. Tanimoto and A. Hosoya, 1993, J. Math. Phys. 35, 4855
- [36] M. Tanimoto, T. Koike and A. Hosoya, 1997, J. Math. Phys. 38, 350
- [37] M. Tanimoto, T. Koike and A. Hosoya, 1997a, J. Math. Phys. 38, 6557
- [38] A. Ashtekar and J. Samuel, 1991, Class.& Quantum Grav. 8, 2191
- [39] W.P. Thurston, 1979, *The Geometry and Topology of 3-manifolds*, Princeton UP
- [40] W.P. Thurston, 1982, Bull. Am. Math. Soc. 6, 357
- [41] J.D. Barrow, R. Juszkiewicz and D.H. Sonoda, 1985, Mon. Not. Roy. astr. Soc., 213, 917
- [42] L. Louko, 1987 Class. Quantum Gravity 4, 581
- [43] D. Eardley, 1974, Comm. Math. Phys. 37, 287
- [44] S.T.C. Siklos, 1978, Comm. Math. Phys. 58, 255